

Painleve property and the first integrals of nonlinear differential equations

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Abstract

Link between the Painleve property and the first integrals of nonlinear ordinary differential equations in polynomial form is discussed. The form of the first integrals of the nonlinear differential equations is shown to determine by the values of the Fuchs indices. Taking this idea into consideration we present the algorithm to look for the first integrals of the nonlinear differential equations in the polynomial form. The first integrals of five nonlinear ordinary differential equations are found. The general solution of one of the fourth ordinary differential equations is given.

Keywords: Nonlinear ordinary differential equation, the Painleve property, the Fuchs indices, the first integral.

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1 Introduction

Finding of the first integrals of the nonlinear ordinary differential equations is one of the most important problem in the theory of nonlinear differential equations. The first integrals allow us to obtain the general solution of nonlinear differential equation in the form of quadratures.

The Painleve property is known to be the necessary condition to integrate nonlinear ordinary differential equation [1–3]. There is the link between

the Painleve property and integrability of nonlinear differential equation. Using the singular manifold method many interesting properties were found [4–8]. We note the Lax pairs and the Backlund transformations for nonlinear partial differential equation [6, 7] and the birational transformations [9, 10] that can be found for nonlinear ordinary differential equations if the nonlinear differential equations have the Painleve property.

There are two aims of this paper. The first aim is to present the link between the Fuchs indices in the expansion of solution in the Laurent series and the first integrals of nonlinear ordinary differential equations. The second aim is taking the Fuchs indices into account to present the algorithm for finding of the first integrals of nonlinear ordinary differential equations that have the Painleve property.

Our approach is illustrated to find the first integrals of the following nonlinear differential equations

$$y_{xxxx} - 10y y_{xx} - 5y_x^2 + 10y^3 + \beta (y_{xx} - 3y^2) + \delta y + \mu = 0 \quad (1.1)$$

$$y_{xxxx} - 10y^2 y_{xx} - 10y y_x^2 + 6y^5 + \beta (y_{xx} - 2y^3) + \delta y + \mu = 0 \quad (1.2)$$

$$y_{xxxx} + 5y_x y_{xx} - 5y^2 y_{xx} - 5y y_x^2 + y^5 - \delta = 0 \quad (1.3)$$

$$y_{xxxx} - 4 \frac{y_x y_{xxx}}{y} + \frac{21 y_x^2 y_{xx}}{2 y^2} - 3 \frac{y_{xx}^2}{y} - 5 \frac{\delta y_{xx}}{y^2} - \frac{9 y_x^4}{2 y^3} + 10 \frac{\delta y_x^2}{y^3} +$$

$$+ \nu y^2 - 2 \frac{\delta^2}{y^3} + \mu = 0 \quad (1.4)$$

$$y_{xxxx} - 2 \frac{y_{xxx} y_x}{y} - \frac{3 y_{xx}^2}{2 y} + 2 \frac{y_{xx} y_x^2}{y^2} - 5 y^2 y_{xx} - \frac{5}{2} y y_x^2 + \frac{5}{2} y^5 -$$

$$-\beta y^3 + \mu y = 0 \quad (1.5)$$

Equation (1.1) can be obtained from the generalized Korteweg – de Vries of the fifth order if we look for solution of this equation in the form of the

travelling wave. Equation (1.1) was found as the special case at the description of the nonlinear wave propagation on the water [11]. This equation can be found at $\beta = 0$ as equation for asymptotic solution of the fourth order analogy for the first Painleve equation [12–15]. Equation (1.1) was studied at $\beta = 0$ in several works [16–18].

Equations (1.2) and (1.3) can be found from the modified Korteweg – de Vries and the Kaup – Kupershmidt equations of the fifth order if we look for solution in the form of the travelling wave. Equation (1.3) was obtained by A P Fordy and J Gibbons in [19]. All these equations are used as equations for the description of the Henon - Heiles model [13, 20, 21]. Equations (1.4) and (1.5) are found as special cases of equations that were obtained recently as the higher order analogies of the Painleve equations [22–25]. All above mentioned equations have the Painleve property.

The outline of this paper is as follows. In section 2 we present the link between the Fuchs indices and the first integrals of nonlinear differential equations in the polynomial form. Section 3 is devoted to the discussion of the algorithm to look for the first integrals of nonlinear ordinary differential equations. In sections 4, 5, 6 and 7 we give examples of finding of the first integrals of the above mentioned nonlinear ordinary differential equations. In section 8 we demonstrate the application of the first integrals to look for the general solution of equation (1.1).

2 Link between the Fuchs indices and the first integrals

Assume we have the nonlinear ordinary differential equation in the polynomial form that passes the Painleve test and assume we are going to find the first integrals to look for the general solution of this equation using quadratures. Taking results of the Painleve analysis into account we have the Fuchs indices and necessary amount of arbitrary constants in the expansion of the general solution in the Laurent series. Our aim is using values of the Fuchs indices to find the general form of the first integrals of nonlinear ordinary differential equations. To understand the link between values of the Fuchs indices and the first integrals of nonlinear ordinary differential equations first of all let us consider two simple examples.

Example 1. Consider the equation

$$E_1[y] = y_{zz} + 6y^2 - ay - b = 0 \tag{2.1}$$

Equation (2.1) has the first integral in the form

$$I_0[y] = y_z^2 + 4y^3 - ay^2 - 2by - C_1 = 0 \quad (2.2)$$

The general solution of equation (2.1) is expressed via the Weierstrass function. The solution of equation (2.1) has one branch in the expansion in the Laurent series and the Fuchs indices $(j_1, j_2) = (-1, 6)$.

Equation possesses the Painleve property and there is the local presentation of the general solution in the form

$$y(x) = -\frac{1}{x^2} + \frac{a}{12} - \left(\frac{a^2}{240} + \frac{b}{10} \right) x^2 + a_6 x^4 + \dots \quad (2.3)$$

where a_6 is arbitrary constant.

Assuming $a = 0$ and $b = 0$ we obtain from equation (2.1) the equation with leading members that is often called as the reduced equation

$$E_1[y] = y_{zz} + 6y^2 = 0 \quad (2.4)$$

The expansion of solution for equation (2.4) in the Laurent series takes the form

$$y(x) = -\frac{1}{x^2} + a_6 x^4 + \dots \quad (2.5)$$

Equation (2.4) has the first integral

$$I_1[y] = y_z^2 + 4y^3 - C_1 = 0 \quad (2.6)$$

Substituting (2.5) into the first integral (2.6) we have

$$I_1[y] = 28a_6 + 4a_6^2 x^6 + 4a_6^3 x^{12} - C_1 = 0 \quad (2.7)$$

From equation (2.7) follows that if we let x tends to 0 then C_1 goes $28a_6$. Therefore arbitrary constant C_1 in the first integral (2.6) is determined by the arbitrary constant a_6 in the expansion of the general solution of equation (2.4) in the Laurent series. However from expansion (2.5) we have $a_6 \simeq y_{xxxx}$. We can believe that the arbitrary constant a_6 corresponds to the pole of the

sixth order because if we use solution $y = -1/x^2$ we obtain the sixth degree. Taking equation (2.7) into account one can note that C_1 also corresponds to the sixth order and the greatest Fuchs index in the expansion of solution in the Laurent series determines the pole order of members in the first integral (2.6).

Example 2. Consider the equation

$$E_2[y] = y_{zzz} - 6y^2 y_z = 0 \quad (2.8)$$

Equation (2.8) possesses the Painleve property and the general solution is expressed via the Jacobi elliptic function. Solution of this equation has the pole of the first order, two branches and the Fuchs indices $(j_1, j_2, j_3) = (-1, 3, 4)$. Equation (2.8) has the first integral

$$I_2[y] = y_{zz} - 2y^3 - C_1 = 0 \quad (2.9)$$

Equation (2.9) also has the first integral

$$I_3[y] = y_z^2 - y^4 - 2C_1 y - C_2 = 0 \quad (2.10)$$

The general solution of equation (2.8) can be written in the form of the Laurent series

$$y(x) = \pm \frac{1}{x} + a_3 x^2 + a_4 x^3 + \dots \quad (2.11)$$

where a_3 and a_4 are arbitrary constants.

It should be noted that one can say about the pole order of the arbitrary constants a_3 and a_4 again because we have $a_3 \simeq y_{xx}$ and $a_4 \simeq y_{xxx}$ from expansion (2.11).

Substituting (2.11) into the first integral (2.8) we obtain

$$\begin{aligned} I_2[y] = & -4a_3 - 6a_3^2 x^3 - 12a_3 a_4 x^4 - 6a_4^2 x^5 - 2a_3^3 x^6 - 6a_3^2 a_4 x^7 - \\ & -6a_3 a_4^2 x^8 - 2a_4^3 x^9 - C_1 = 0 \end{aligned} \quad (2.12)$$

Letting $x \rightarrow 0$ in (2.12) we obtain $C_1 = -4a_3$. We have again from equation (2.12) that the arbitrary constant C_1 in the first integral (2.9) is determined by the arbitrary constant a_3 that corresponds to the Fuchs indices.

However the arbitrary constant a_3 corresponds to the pole of the third order because $a_3 \simeq y_{xx}$. We have that the arbitrary constant C_1 also corresponds to the pole of the third order and we can observe that members of the left hand side in the first integral (2.9) has the pole of the third order.

Substituting (2.11) into the first integral (2.10) at $C_1 = -4a_3$ we have

$$\begin{aligned} I_3[y] = & -C_2 - 10a_4 + 6a_3^2x^2 + 8a_3a_4x^3 + 3a_4^2x^4 - 4a_3^3x^5 - \\ & -12a_3^2a_4x^6 - 12a_3a_4^2x^7 - (a_3^4 + 4a_4^3)x^8 - 4a_3^3a_4x^9 - \\ & -6a_3^2a_4^2x^{10} - 4a_3a_4^3x^{11} - a_4^4x^{12} = 0 \end{aligned} \quad (2.13)$$

Letting $x \longrightarrow 0$ in (2.13) we obtain $C_2 = -10a_4$. We can see that the arbitrary constant C_2 in the first integral (2.10) is determined by the arbitrary constant a_4 . The pole order corresponds to the arbitrary constant a_4 equal to four because $a_4 \simeq y_{xxx}$. We observe again that the members of the left hand side in the first integral (2.10) has the fourth order and corresponds to the Fuchs index.

These examples allow us to formulate the following theorem.

Theorem 2.1. *Let the nonlinear differential equation in a polynomial form posses the Painleve property then the pole order of members in the first integrals of the reduced equation is determined by the Fuchs indices.*

Proof. Consider the nonlinear ordinary differential equation in the polynomial form that possesses the Painleve property and consequently the reduced equation also has the Painleve property.

We also consider the autonomous ordinary differential equations in this paper that posses the Painleve property and have the first integrals in the polynomial form. We know many nonlinear ordinary differential equations (for example the Painleve equations and the higher order Painleve equations) that have not got any first integrals in a polynomial form.

Solution of equation studied can be presented in the Lorent series. Some coefficients ($n - 1$, where n is the order of equation) of the expansion in the Laurent series are arbitrary constants. Substituting $y(x) = B_0/x^p$ into the leading members of equation studied we have branches of solutions, that correspond to different values (B_0, p) . As consequence of finding (B_0, p) the equation with leading members is satisfied by solution $y(x) = B_0/x^p$.

If equation has the first integral in the polynomial form then the reduced equation has the first integral in the polynomial form as well. Substituting

$y(x) = B_0/x^p$ into the first integral for the reduced equation we have that is solution of the first integral when arbitrary constant equals zero.

Substituting the Laurent series for the general solution of equation with the leading members into the corresponding first integral we obtain the arbitrary constant in the first integral that is determined by the arbitrary constants from the Laurent series. Therefore the arbitrary constant in the first integral of the nonlinear ordinary differential equation with leading members is determined by the arbitrary constants in the expansion of solution in the Laurent series.

However the arbitrary constants in the expansion of the solution in the Laurent series correspond to the Fuchs indices at investigation of equation on the Painleve property. Taking the pole orders of arbitrary constants into consideration (that are determined by the Fuchs indices) we determine the pole order of arbitrary constant in the first integral and consequently the pole order of members in the first integral. \square

This theorem allows us to determine the form of the first integrals of the reduced equation (equation with the leading members) and to find the first integrals for the more general equation in the polynomial form.

3 Algorithm to look for the first integrals of nonlinear differential equations

Let us use the results of the theorem 2.1 to look for the first integrals of nonlinear ordinary differential equations. Without loss of the generality we consider the fourth order ordinary differential equations. Denote $y = y_0$, $y_x = y_1$, $y_{xx} = y_2$, $y_{xxx} = y_3$, $y_{xxxx} = y_4$ to simplify calculations.

Let us assume there is the nonlinear differential equation of the fourth order in the form

$$y_4 = E(y_0, y_1, y_2, y_3) \quad (3.1)$$

Let us suppose there is the first integrals of this equation

$$P(y_0, y_1, y_2, y_3) = K_1 \quad (3.2)$$

where K_1 and further K_2 , K_3 and K_4 are arbitrary constants.

Let us obtain the equation for finding of the first integrals of nonlinear ordinary differential equations. By definition of the first integral of equation (3.1) we have

$$\sum_{n=0}^3 y_{n+1} \frac{\partial P}{\partial y_n} = Q(y_0, y_1, y_2, y_3, y_4) (y_4 - E(y_0, y_1, y_2, y_3)) \quad (3.3)$$

where $Q(y_0, y_1, y_2, y_3, y_4)$ is arbitrary expression depending on variable y_0 and its derivatives.

Taking into account (3.1) we get from equation (3.3)

$$\sum_{n=0}^2 y_{n+1} \frac{\partial P}{\partial y_n} + E(y_0, y_1, y_2, y_3) \frac{\partial P}{\partial y_3} = 0 \quad (3.4)$$

The last equation is the basic equation that can be used to look for the first integrals of the nonlinear differential equation. Substituting $E(y_0, y_1, y_2, y_3)$ from equation (3.1) into equation (3.4) we have equation for finding of the first integrals of equation (3.1). To look for the first integral of equation (3.1) we have to have additional information about the form of P . This information can be extracted from the Fuchs indices of equation (3.1). Taking the Fuchs indices into account we can determine the pole order of the first integral and we can write the polynomial with unknown coefficients. Substituting the polynomial with unknown coefficients into equation (3.4) we can find unknown coefficients solving the algebraic linear equations and obtain the first integral.

Our approach has the following steps: 1) The Painleve test of origin equation on the Painleve property and finding the Fuchs indices; 2) Determination of the reduced equation from the origin equation. This equation can be found from the origin equation if we substitute $y_0 = B_0/x^p$ into the origin equation; 3) Writing the polynomial with unknown coefficients taking into account values of the Fuchs indices; 4) Determination of unknown coefficients of polynomial and writing the first integral of the reduced equation; 5) Writing additional polynomials with unknown coefficients that correspond to the origin equation; 6) Determination of unknown coefficients of the additional polynomials and finding the first integral of the origin equation.

4 The first integrals of equation (1.1)

Let us apply our approach to look for the first integrals of the equation

$$y_{xxxx} - 10y y_{xx} - 5y_x^2 + 10y^3 + \beta (y_{xx} - 3y^2) + \delta y + \mu = 0 \quad (4.1)$$

This equation possesses the Painleve property. The general solution has two branches with $(B_0, p) = (2, 2)$, $(B_0, p) = (6, 2)$ and the following Fuchs indices $(m_1, m_2, m_3, m_4) = (-1, 2, 5, 8)$ and $(m_1, m_2, m_3, m_4) = (-3, -1, 8, 10)$.

The reduced equation can be found from equation (4.1) at $\beta = 0$, $\delta = 0$, $\mu = 0$ and takes the form

$$y_{xxxx} - 10y y_{xx} - 5y_x^2 + 10y^3 = 0 \quad (4.2)$$

Let us find the first integrals of equation (4.2). The greatest Fuchs indices are equal to 8 and 10. One can expect that the pole order of members for the first integral of equation (4.2) is equal to 8 and the pole order of members for another first integral is equal to 10. Let us write two polynomial with unknown coefficients that correspond to the poles order 8 and 10.

We denote the polynomial $P_k^{(j)}$ as polynomial with the k-th pole order of members and the j-th pole order for solution of the nonlinear ordinary differential equations.

The first polynomial to look for the first integral of equation (4.2) has the form

$$P_8^{(2)} = A_0 y_3 y_1 + A_1 y_2^2 + A_2 y_2 y_0^2 + A_3 y_1^2 y_0 + A_4 y_0^4 \quad (4.3)$$

where A_0, \dots, A_4 are unknown coefficients that should be found. One can note that members in the polynomial (4.3) has the eighth order.

The second polynomial to search the first integral of equation (4.2) takes the form

$$P_{10}^{(2)} = A_0 y_3^2 + A_1 y_3 y_1 y_0 + A_2 y_2^2 y_0 + A_3 y_2 y_1^2 + A_4 y_2 y_0^3 + A_5 y_1^2 y_0^2 + A_6 y_0^5 \quad (4.4)$$

Substituting polynomial (4.3) into equation (3.4) and solving algebraic equations for the unknown coefficients A_0, \dots, A_4 we have the first integral for equation (4.2) in the form

$$I_4[y] = y_x y_{xxx} - \frac{1}{2} y_{xx}^2 - 5 y y_x^2 + \frac{5}{2} y^4 - C_1 = 0 \quad (4.5)$$

The expansion of the general solution of equation (4.2) in the Laurent series for the first branch takes the form

$$y(x) = \frac{2}{x^2} + a_2 - \frac{3}{2} a_2^2 x^2 + a_5 x^3 - \frac{5}{2} a_2^3 x^4 + \frac{3}{4} a_2 a_5 x^5 + a_8 x^6 + \dots \quad (4.6)$$

Substituting solution (4.6) into the first integral (4.5) and letting $x \rightarrow 0$ we obtain

$$I_4[y_1] = -567 a_2^4 - 648 a_8 - C_1 = 0 \quad (4.7)$$

Equality (4.7) confirms our choice of the polynomial (4.3) and the theorem 2.1.

Later we need the following polynomials

$$P_0^{(2)} = A_0, \quad P_2^{(2)} = A_0 y_0, \quad P_3^{(2)} = A_0 y_1, \quad P_4^{(2)} = A_0 y_2 + A_1 y_0^2, \quad (4.8)$$

$$P_5^{(2)} = A_0 y_3 + A_1 y_1 y_0, \quad P_6^{(2)} = A_0 y_4 + A_1 y_2 y_0 + A_2 y_1^2 + A_3 y_0^3$$

Adding to the first integral (4.5) three polynomials with unknown coefficients in the form $\beta P_4^{(2)} + \delta P_3^{(2)} + \mu P_2^{(2)}$ we obtain the following first integral of equation (4.1)

$$y_x y_{xxx} - \frac{1}{2} y_{xx}^2 - 5 y y_x^2 + \frac{5}{2} y^4 + \frac{1}{2} \beta (y_x^2 - 2 y^3) + \frac{1}{2} \delta y^2 + \mu y = K_1 \quad (4.9)$$

Substituting polynomial (4.4) into equation (??) and finding A_0, \dots, A_6 we obtain another first integral for the equation (4.2) in the form

$$y_{xxxx}^2 - 12 y y_x y_{xxx} - 4 y y_{xx}^2 + 2 y_x^2 y_{xx} + 20 y^3 y_{xx} + 30 y^2 y_x^2 - 24 y^5 - C_2 = 0 \quad (4.10)$$

Adding to (4.10) $\beta P_6 + \delta P_5 + \mu P_4$ and substituting them into equation (??) again we have the first integral in the form

$$y_{xxxx}^2 - 12 y y_x y_{xxx} - 4 y y_{xx}^2 + 2 y_x^2 y_{xx} + 20 y^3 y_{xx} + 30 y^2 y_x^2 - 24 y^5 + \beta (y_{xx} - 3 y^2)^2 + \delta (2 y y_{xx} - 4 y^3 - y_x^2) + 2 \mu (y_{xx} - 3 y^2) = K_2 \quad (4.11)$$

The first integrals (4.9) and (4.11) can be used to look for the general solution of equation (4.1).

5 The first integrals of equation (1.2)

Let us find the first integrals of the equation

$$y_{xxxx} - 10y^2 y_{xx} - 10y y_x^2 + 6y^5 + \beta (y_{xx} - 2y^3) + \delta y + \mu = 0 \quad (5.1)$$

This equation passes the Painleve test. Solution of (5.1) has the pole of the first order and four branches with $(B_0, p) = (1, 1)$, $(B_0, p) = (-1, 1)$, $(B_0, p) = (2, 1)$ and $(B_0, p) = (-2, 1)$. These branches have two collections of the Fuchs indices: $(j_1, j_2, j_3, j_4) = (-1, 2, 3, 6)$ and $(j_1, j_2, j_3, j_4) = (-3, -1, 6, 8)$. The greatest Fuchs indices are equals to six and eight. Therefore we can expect that members of the first integrals have poles of the sixth and eighth order.

The reduced equation can be found from equation (5.1) at $\beta = 0$, $\delta = 0$, $\mu = 0$ and takes the form

$$y_{xxxx} - 10y^2 y_{xx} - 10y y_x^2 + 6y^5 = 0 \quad (5.2)$$

In fact one of the first integral of equation (5.2) can be found if we take the pole of the sixth order into account and multiply equation (5.2) on y_x but we are going to use our approach and to construct the polynomial with unknown coefficients that has members with pole of the sixth order. This polynomial can be written in the form

$$\begin{aligned} P_6^{(1)} = & a_0 y_1 y_3 + a_1 y_3 y_0^2 + a_2 y_2^2 + a_3 y_2 y_1 y_0 + a_4 y_2 y_0^3 a_5 y_1^3 + \\ & + a_6 y_1^2 y_0^2 + a_7 y_1 y_0^4 + a_8 y_0^6 \end{aligned} \quad (5.3)$$

where a_0, \dots, a_8 are unknown coefficients that should be found.

Substituting (5.3) into equation (3.4) and solving the system of the linear algebraic equations for coefficients a_0, \dots, a_8 we have the first integral of equation (5.2) in the form

$$I_5[y] = y_1 y_3 - \frac{1}{2} y_2^2 - 5 y^2 y_1^2 + y_0^6 - C_1 = 0 \quad (5.4)$$

The expansion of the solution of equation (5.2) in the Laurent series of the first branch takes the form

$$y(x) = \frac{1}{x} + a_2 x + a_3 x^2 + \frac{5}{2} a_2^2 x^3 + \frac{5}{3} a_2 a_3 x^4 + a_6 x^5 + \dots \quad (5.5)$$

Substituting the expansion (5.5) into the first integral (5.4) and letting $x \rightarrow 0$ we have the equality

$$I_5[y_1] = 210 a_2^3 + 28 a_3^2 - 84 a_6 - C_1 = 0 \quad (5.6)$$

This equality again gives the confirmation of the theorem 2.1.

Later we are also going to use the following polynomials

$$P_1^{(1)} = a_0 y_0, \quad P_2^{(1)} = a_0 y_1 + a_1 y_0^2, \quad P_3^{(1)} = a_0 y_2 + a_1 y_1 y_0 + a_2 y_0^3, \quad (5.7)$$

$$P_4^{(1)} = a_0 y_3 + a_1 y_2 y_0 + a_2 y_1^2 + a_3 y_1 y_0^2 + a_4 y_0^4 \quad (5.8)$$

$$P_5^{(1)} = a_0 y_4 + a_1 y_3 y_0 + a_2 y_2 y_1 + a_3 y_1^2 y_0 + a_4 y_1 y_0^3 + a_5 y_0^5 \quad (5.9)$$

$$P_5^{(1)} = a_0 y_4 + a_1 y_3 y_0 + a_2 y_2 y_1 + a_3 y_1^2 y_0 + a_4 y_1 y_0^3 + a_5 y_0^5 \quad (5.10)$$

$$P_7^{(1)} = a_0 y_3 y_2 + a_1 y_3 y_1 y_0 + a_2 y_2^2 y_0 + a_3 y_2 y_1^2 + a_4 y_2 y_1 y_0^2 + a_5 y_2 y_0^4 + \quad (5.11)$$

$$+ a_6 y_1^3 y_0 + a_7 y_1^2 y_0^3 + a_8 y_1 y_0^5 + a_9 y_0^7$$

Taking parameters β , δ and μ into account and adding to (5.4) the polynomials $\beta P_4^{(1)} + \delta P_2^{(1)} + \mu P_1^{(1)}$ we have the first integral of equation (5.1) in the form

$$y_x y_{xxx} - \frac{1}{2} y_{xx}^2 - 5 y^2 y_x^2 + y^6 + \frac{1}{2} \beta (y_x^2 - y^4) + \frac{1}{2} \delta y^2 + \mu y = K_1 \quad (5.12)$$

To find another first integral we use the polynomial with members of the eighth order pole

$$P_8^{(1)} = a_0 y_3^2 + a_1 y_3 y_2 y_0 + a_3 y_3 y_1^2 + a_4 y_3 y_1 y_0^2 + a_5 y_2^2 y_1 + a_6 y_2^2 y_0^2 + \quad (5.13)$$

$$+ a_7 y_2 y_1^2 y_0 + a_8 y_2 y_1 y_0^3 + a_9 y_2 y_0^5 + a_{10} y_1^4 + a_{11} y_1^3 y_0^2 +$$

$$+ a_{12} y_1^2 y_0^4 + a_{13} y_1 y_0^6 + a_{14} y_0^8$$

Substituting polynomial (5.13) into equation (3.4) and taking equation (5.2) into account we have another first integral of equation (5.2)

$$\begin{aligned}
& y_3^2 - 12 y_3 y_1 y_0^2 - 4 y_2^2 y_0^2 + 4 y_2 y_1^2 y_0 + 12 y_2 y_0^5 - y_1^4 + \\
& + 30 y_1^2 y^4 - 9 y_0^8 = C_2
\end{aligned} \tag{5.14}$$

Using (5.14) and adding $\beta P_6^{(1)} + \delta P_4^{(1)} + \mu P_3^{(1)}$ we have the first integral of equation (5.1) in the form

$$\begin{aligned}
& y_{xxx}^2 - 12 y^2 y_x y_{xxx} - 4 y^2 y_{xx}^2 + 4 y y_x^2 y_{xx} + 12 y^5 y_{xx} - \\
& - y_x^4 + 30 y^4 y_x^2 - 9 y^8 + \beta (y_{xx} - 2 y^3)^2 + \\
& + \delta (2 y y_{xx} - y_x^2 - 3 y^4) + 2 \mu (y_{xx} - 2 y^3) = K_2
\end{aligned} \tag{5.15}$$

Equations (4.1) and (5.1) can be written as the Hamiltonian systems and the first integrals found enough to find the general solutions of these equations.

6 The first integrals of equation (1.3)

Let us find the first integrals of the equation

$$y_{xxxx} + 5 y_x y_{xx} - 5 y^2 y_{xx} - 5 y y_x^2 + y^5 - \delta = 0 \tag{6.1}$$

This equation passes the Painleve test. There are four branches of solutions with $(B_0, p) = (1, 1)$, $(B_0, p) = (-3, 1)$, $(B_0, p) = (-2, 1)$, $(B_0, p) = (4, 1)$ and three collections of the Fuchs indices that are $(j_1, j_2, j_3, j_4) = (-1, 2, 3, 6)$, $(j_1, j_2, j_3, j_4) = (-1, 2, 6, 7)$ and $(j_1, j_2, j_3, j_4) = (-7, -1, 6, 12)$. The first and third branches of solutions have the same Fuchs indices.

Equation with leading members takes the form

$$y_{xxxx} - 5 y^2 y_{xx} + 5 y_x y_{xx} - 5 y y_x^2 + y^5 = 0 \tag{6.2}$$

There are the Fuchs indices equal six and we can find one of the first integral of equation (6.2) using the polynomial $P_6^{(1)}$ from the previous section

but we can also obtain this first integral if we multiply (6.2) on y_x and integrating this equality. We have the first integral of equation (6.2) in the form

$$I_6[y] = y_3 y_1 - \frac{1}{2} y_2^2 + \frac{5}{3} y_1^3 - \frac{5}{2} y_1^2 y_0^2 + \frac{1}{6} y_0^6 - C_1 = 0 \quad (6.3)$$

By analogy we can find the first integral of equation (6.1)

$$y_x y_{xxx} - \frac{1}{2} y_{xx}^2 - \frac{5}{2} y^2 y_x^2 + \frac{1}{6} y^6 + \frac{5}{3} y^3 - \delta y = K_1 \quad (6.4)$$

Substituting the expansions in the Laurent series for the four branches of the solution into the first integral (6.3) and letting $x \rightarrow 0$ we have equalities

$$I_6[y_1] = \frac{21}{4} a_2^3 - 12 a_3^2 - 84 a_6 - C_1 = 0 \quad (6.5)$$

$$I_6[y_2] = C_1 + 168 a_6 = 0 \quad (6.6)$$

$$I_6[y_3] = -816 a_2^3 + 108 a_3^2 + 168 a_6 - C_1 = 0 \quad (6.7)$$

$$I_6[y_4] = 2184 a_6 - C_1 = 0 \quad (6.8)$$

We have again the confirmation of the theorem 2.1 for the first integral (6.3) of equation (6.2).

To find another first integral we use the polynomial with members that

have the poles of the twelfth order

$$\begin{aligned}
P_{12}^{(1)} = & a_0 y_3^3 + (a_1 y_2 y_0 + a_2 y_1^2 + a_3 y_1 y_0^2 + a_4 y_0^4) y_3^2 + \\
& + (a_5 y_0^2 + a_6 y_1) y_3 y_2^2 + (a_7 y_1^2 y_0 + a_8 y_1 y_0^3 + a_9 y_0^5) y_3 y_2 + \\
& + (a_{10} y_1^4 + a_{11} y_1^3 y_0^2 + a_{12} y_1^2 y_0^4 + a_{13} y_1 y_0^6 + a_{14} y_0^8) y_3 + b_0 y_2^4 + \\
& + (b_1 y_1 y_0 + b_2 y_0^3) y_2^3 + (b_3 y_1^3 + b_4 y_1^2 y_0^2 + b_5 y_1 y_0^4 + b_6 y_0^6) y_2^2 + \\
& + (b_7 y_1^4 y_0 + b_8 y_1^3 y_0^3 + b_9 y_1^2 y_0^5 + b_{10} y_1 y_0^7 + b_{11} y_0^9) y_2 + c_0 y_1^6 + \\
& + (c_1 y_1^4 + c_2 y_1^3 y_0^2 + c_3 y_1^2 y_0^4 + c_4 y_1 y_0^6 + c_5 y_0^8) y_1 y_0^2 + c_6 y_0^{12}
\end{aligned} \tag{6.9}$$

where a_0, \dots, a_{14} , b_0, \dots, b_{11} and c_0, \dots, c_6 are unknown coefficients that should be found. Substituting this polynomial into equation (3.4) we have the first integral of equation (6.2) in the form

$$\begin{aligned}
& y_3^3 + \left(\frac{3}{2} y_0^4 - 9 y_0^2 y_1 \right) y_3^2 + F_1(y_0, y_1, y_2) y_3 - \frac{15}{8} y_2^4 + 2 y_0^3 y_2^3 + \\
& + F_2(y_0, y_1) y_2^2 + F_3(y_0, y_1) y_2 - \frac{22}{3} y_1^6 + \frac{35}{2} y_0^2 y_1^5 + \frac{45}{8} y_0^4 y_1^4 - \\
& - \frac{157}{6} y_0^6 y_1^3 + \frac{19}{4} y_0^8 + 3 y_0^{10} y_1 - \frac{17}{24} y_0^{12} = C_2
\end{aligned} \tag{6.10}$$

where

$$\begin{aligned}
F_1(y_0, y_1, y_2) = & \left(\frac{15}{2} y_1 - 3 y_0^2 \right) y_2^2 + 3 (y_1^2 - 2 y_0^2 y_1 + y_0^4) y_0 y_2 - \\
& - 7 y_1^4 - \frac{9}{2} y_0^2 y_1^3 + 30 y_0^4 y_1^2 - \frac{17}{2} y_0^6 y_1,
\end{aligned}$$

$$F_2(y_0, y_1) = \frac{25}{2} y_1^3 + 15 y_0^4 y_1 - \frac{117}{4} y_0^2 y_1^2 - \frac{13}{4} y_0^6,$$

$$F_3(y_0, y_1) = 9y_0 y_1^4 - 30y_0^3 y_1^3 + 36y_0^5 y_1^2 - 18y_0^7 y_1 + 3y_0^9$$

Taking into account the first integral (6.10) and polynomials $\delta P_7^{(1)} + \delta^2 P_2^{(1)}$ we have another first integral in the form

$$\begin{aligned} & y_{xxx}^3 + \left(\frac{3}{2}y^4 - 9y^2 y_x \right) y_{xxx}^2 + F_4(y, y_x, y_{xx}) y_{xxx} - \frac{15}{8} y_{xx}^4 + 2y^3 y_{xx}^3 + \\ & + F_5(y, y_x) y_{xx}^2 + F_6(y, y_x) y_{xx} - \frac{22}{3} y_x^6 + \frac{35}{2} y^2 y_x^5 + \frac{45}{8} y^4 y_x^4 - \\ & - \left(\frac{157}{6} y^6 - 7\delta y \right) y_x^3 + \left(\frac{19}{4} y^8 - \frac{27}{2} \delta y^3 \right) y_x^2 + \\ & + F_7(y) y_x - \frac{17}{24} y^{12} - \frac{9}{2} \delta^2 y^2 = K_2 \end{aligned} \tag{6.11}$$

where

$$\begin{aligned} F_4(y, y_x, y_{xx}) &= \left(\frac{15}{2} y_x - 3y^2 \right) y_{xx}^2 + 3(y_x^2 - 2y^2 y_x + y^4) y y_{xx} - \\ & - 3\delta y_{xx} - 7y_x^4 - \frac{9}{2} y^2 y_x^3 + 30y^4 y_x^2 - \left(\frac{17}{2} y^6 - 9\delta y \right) y_x, \\ F_5(y, y_x) &= \frac{25}{2} y_x^3 + 15y^4 y_x - \frac{117}{4} y^2 y_x^2 - \frac{13}{4} y^6 - \frac{9}{2} \delta y, \\ F_6(y, y_x) &= 9y y_x^4 - 30y^3 y_x^3 + 36y^5 y_x^2 - 9\delta y_x^2 + \\ & + 18\delta y^2 y_x - 18y^7 y_x + 3y^9 - 3\delta y^4, \\ F_7(y) &= 3y^{10} - 6\delta y^5 + 3\delta^2 \end{aligned}$$

We emphasize that the second branch of solution of equation (6.1) has the Fuchs index equal seven but we do not have this value for the determination of the arbitrary constant C_1 in the first integrals found. New first integral can be tolerated with this Fuchs index that can be found if we use the polynomial with members of greater pole order .

7 The first integrals of equations (1.4) and (1.5)

Equations (1.1), (1.2) and (1.3) have the polynomial form and we can observe that the Fuchs indices in the expansion of solutions in the Laurent series determine the first integrals of nonlinear differential equations. There is question about other equations with this property. Later we are going to study the problem of finding of the first integrals in the case when nonlinear differential equations have the non—polynomial form. With this aim let us find the first integrals of equation

$$y_{xxxx} - 4 \frac{y_x y_{xxx}}{y} + \frac{21 y_x^2 y_{xx}}{2 y^2} - 3 \frac{y_{xx}^2}{y} - 5 \frac{\delta y_{xx}}{y^2} - \frac{9 y_x^4}{2 y^3} + 10 \frac{\delta y_x^2}{y^3} + \nu y^2 - 2 \frac{\delta^2}{y^3} + \mu = 0 \quad (7.1)$$

Equation (7.1) passes the Painleve test. Solution has the first order pole. We get five branches of solutions but two collections of the Fuchs indices that are $(j_1, j_2, j_3, j_4) = (-1, 1, 2, 4)$ and $(j_1, j_2, j_3, j_4) = (-3, -1, 4, 6)$.

Consider the reduced equation

$$y_{xxxx} - 4 \frac{y_x y_{xxx}}{y} + \frac{21 y_x^2 y_{xx}}{2 y^2} - 3 \frac{y_{xx}^2}{y} - \frac{9 y_x^4}{2 y^3} = 0 \quad (7.2)$$

Using values for the Fuchs indices we can find the form of the first integrals. We take the expression in the form $P_8^{(1)}/y_0^4$ to obtain the value that corresponds to the order four of the Fuchs indices. We found the first integral of equation (7.2) in the form

$$\frac{y_x y_{xxx}}{y^2} - \frac{1}{2} \frac{y_{xx}^2}{y^2} - 2 \frac{y_x^2 y_{xx}}{y^3} + \frac{9}{8} \frac{y_x^4}{y^4} = C_1 \quad (7.3)$$

Using additional polynomials $(\nu P_5^{(1)} + \delta P_4^{(1)} + \mu P_3^{(1)} + \delta^2 P_0^{(1)})/y_0^4$ we obtain the first integral of the equation (7.1)

$$\frac{y_x y_{xxx}}{y^2} - \frac{1}{2} \frac{y_{xx}^2}{y^2} - 2 \frac{y_x^2 y_{xx}}{y^3} + \frac{9}{8} \frac{y_x^4}{y^4} - \frac{5}{2} \frac{\delta y_x^2}{y^4} + \frac{1}{2} \frac{\delta^2}{y^4} - \frac{\mu}{y} + \nu y = K_1 \quad (7.4)$$

Taking the value of the Fuchs indices equal six into account and using the expression with unknown coefficients in the form $P_{12}^{(1)}/y_0^6$ we have the first integral of the equation (7.2) in the form

$$\frac{y_{xxx}^2}{y^2} - 6 \frac{y_x y_{xx} y_{xxx}}{y^3} + 3 \frac{y_x^3 y_{xxx}}{y^4} + 9 \frac{y_x^2 y_{xx}^2}{y^4} - 9 \frac{y_x^4 y_{xx}}{y^5} + \frac{9}{4} \frac{y_x^6}{y^6} = C_2 \quad (7.5)$$

Using equation (7.5) and the additional polynomials $(\nu P_9^{(1)} + \delta P_8^{(1)} + \mu P_7^{(1)} + \delta \nu P_5^{(1)} + \delta^2 P_4^{(1)} + \delta \mu P_3^{(1)} + \delta^3 P_0^{(1)})/y_0^6$ we get another first integral of equation (7.1) in the form

$$\begin{aligned} & \frac{y_{xxx}^2}{y^2} - 6 \frac{y_x y_{xx} y_{xxx}}{y^3} + 3 \frac{y_x^3 y_{xxx}}{y^4} + 9 \frac{y_x^2 y_{xx}^2}{y^4} - 9 \frac{y_x^4 y_{xx}}{y^5} + \frac{9}{4} \frac{y_x^6}{y^6} - \\ & - 2 \delta \left(3 \frac{y_x y_{xxx}}{y^4} + \frac{y_{xx}^2}{y^4} - 10 \frac{y_x^2 y_{xx}}{y^5} + \frac{19}{4} \frac{y_x^4}{y^6} \right) - 2 \frac{\delta^3}{y^6} + 2 \frac{\delta \mu}{y^3} + 6 \frac{\delta \nu}{y} + \\ & + 2 \nu y_{xx} - 3 \frac{\nu y_x^2}{y} + 2 \frac{\mu y_{xx}}{y^2} - \frac{\mu y_x^2}{y^3} - \delta^2 \left(4 \frac{y_{xx}}{y^5} - 11 \frac{y_x^2}{y^6} \right) = K_2 \quad (7.6) \end{aligned}$$

Now let us find the first integrals of the equation

$$\begin{aligned} & y_{xxxx} - 2 \frac{y_{xxx} y_x}{y} - \frac{3 y_{xx}^2}{2 y} + 2 \frac{y_{xx} y_x^2}{y^2} - 5 y^2 y_{xx} - \frac{5}{2} y y_x^2 + \frac{5}{2} y^5 - \\ & - \beta y^3 + \mu y = 0 \end{aligned} \quad (7.7)$$

Equation passes the Painleve test and has four branches of solution. These branches have two collections of the Fuchs indices that are $(j_1, j_2, j_3, j_4) = (-1, 1, 3, 5)$ and $(j_1, j_2, j_3, j_4) = (-2, -1, 5, 6)$. We can expect that the first integrals of the reduced equation

$$y_{xxxx} - 2 \frac{y_{xxx} y_x}{y} - \frac{3 y_{xx}^2}{2 y} + 2 \frac{y_{xx} y_x^2}{y^2} - 5 y^2 y_{xx} - \frac{5}{2} y y_x^2 + \frac{5}{2} y^5 = 0 \quad (7.8)$$

can be obtained in the form of the expressions that correspond to the Fuchs indices equal five and six. Taking $P_7^{(1)}/y_0^2$ into account we obtain the first integral of equation (7.8) in the form

$$\frac{y_x y_{xxx}}{y} - \frac{1}{2} \frac{y_{xx}^2}{y} - \frac{y_x^2 y_{xx}}{y^2} - \frac{5}{2} y y_x^2 + \frac{1}{2} y^5 = C_1 \quad (7.9)$$

Using the first integral (7.9) and the additional polynomials $(\beta P_5^{(1)} + \mu P_3^{(1)})/y_0^2$ we get the first integral of equation (7.7) in the form

$$\frac{y_x y_{xxx}}{y} - \frac{1}{2} \frac{y_{xx}^2}{y} - \frac{y_x^2 y_{xx}}{y^2} - \frac{5}{2} y y_x^2 + \frac{1}{2} y^5 - \frac{1}{3} \beta y^3 + \mu y = K_1 \quad (7.10)$$

To find another first integral we take the expression $P_{10}^{(1)}/y_0^4$ into consideration. We obtain

$$\begin{aligned} \frac{y_{xxx}^2}{y^2} - 2 \frac{y_x y_{xx} y_{xxx}}{y^3} - 8 y_{xxx} y_x - \frac{1}{3} \frac{y_{xx}^3}{y^3} + \frac{y_x^2 y_{xx}^2}{y^4} + 11 \frac{y_x^2 y_{xx}}{y} - \\ - y_{xx}^2 + 5 y^3 y_{xx} + 10 y^2 y_x^2 - \frac{10}{3} y^6 = C_2 \end{aligned} \quad (7.11)$$

Taking the first integral (7.11) and polynomials $(\mu P_6^{(1)} + \beta P_4^{(1)})/y_0^4$ into account we get another first integral of equation (7.7) in the form

$$\begin{aligned} \frac{y_{xxx}^2}{y^2} - 2 \frac{y_x y_{xx} y_{xxx}}{y^3} - \frac{1}{3} \frac{y_{xx}^3}{y^3} + \frac{y_x^2 y_{xx}^2}{y^4} + 11 \frac{y_x^2 y_{xx}}{y} - \\ - 8 y_{xxx} y_x - y_{xx}^2 + 5 y^3 y_{xx} + 10 y^2 y_x^2 - \frac{10}{3} y^6 + 2 \frac{\mu y_{xx}}{y} - 4 \mu y^2 - \\ - 2 \beta y y_{xx} + 2 \beta y_x^2 + 2 \beta y^4 = K_2 \end{aligned} \quad (7.12)$$

We have obtained the first integrals for several nonlinear ordinary differential equations. All these first integrals have members with the pole orders that are determined via the Fuchs indices.

8 General solution of equation (1.1)

Let us find the general solution of equation (1.1) using the first integrals (4.9) and (4.11). At $\beta = 0$ equation was studied before [16–18]. Here let us present the general solution in the case $\beta \neq 0$. Denote new variables

$$M = y_{xx} - 3 y^2 - \frac{1}{2} \delta, \quad (8.1)$$

$$N = y y_{xx} - \frac{1}{2} y_x^2 - 3 y^3 - \frac{1}{2} \mu \quad (8.2)$$

then we can write the first integrals (4.9) and (4.11) in the form

$$y_x M_x - \left(y^2 + \frac{1}{2} \delta - \beta y \right) M - (\beta + 2y) N - \frac{1}{2} M^2 + \frac{1}{2} \beta \delta y = K_1 \quad (8.3)$$

$$M_x^2 + \beta M^2 - 4 M N + \beta \delta M = K_2, \quad (8.4)$$

Let $P(t)$ be the curve of the genus two in the form

$$P(t) = t^5 + m_0 t^4 + m_1 t^3 + m_2 t^2 + m_3 t + m_4$$

where m_0, \dots, m_4 are unknown coefficients.

Let us assume that y have the form

$$y = \frac{1}{2}(u(x) + v(x) - \beta), \quad M(x) = \frac{1}{2}u(x)v(x) \quad (8.5)$$

where $u(x)$ and $v(x)$ satisfy the following equations

$$(u - v) u_x = \sqrt{P(u)}, \quad (u - v) v_x = -\sqrt{P(v)} \quad (8.6)$$

Substituting (8.5) and (8.6) into the first integrals (8.3) and (8.4) we obtain $P(t)$ in the form

$$P(t) = t^5 - 3\beta t^4 + (3\beta^2 + 2\delta) t^2 - 4\mu t^2 + 2(\beta^2 \delta + 4K_1) t + 4K_2 \quad (8.7)$$

The system of equations (8.6) can be written in the form

$$I_0(u(x)) + I_0(v(x)) = K_3, \quad I_1(u(x)) + I_1(v(x)) = x + K_4 \quad (8.8)$$

where

$$I_0(u(x)) = \int_{\infty}^{u(x)} \frac{dt}{\sqrt{P(t)}}, \quad I_1(u(x)) = \int_{\infty}^{u(x)} \frac{t dt}{\sqrt{P(t)}}, \quad (8.9)$$

The general solution of equation (1.1) is expressed via the hyperelliptic functions taking equations (8.7) and (8.8) into account.

9 Conclusion

In this paper we considered the problem of finding of the first integral for nonlinear ordinary differential equations. We have observed that arbitrary constant of the first integral is determined by the arbitrary constants in the expansion of the general solution in the Laurent series. However the arbitrary constants for the general solution in the Laurent series have the pole order that correspond to the Fuchs indices. We have found the link between the pole order of the members in the first integrals and the Fuchs indices of the expansion in the Laurent series. This observation allowed us to present the algorithm to look for the first integrals of nonlinear differential equations that posses the Painleve property. Algorithm was applied to look for the first integrals of five nonlinear ordinary differential equations.

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